

# ELASTIC INSTABILITY ANALYSIS IN BRIDGE PILES PARTIALLY EMBEDDED IN CLAY OR SAND, APPLYING THE METHOD OF GALERKIN

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**Abstract.** This work is aimed to calculate the critical loads that could cause elastic instability in slender piles used in modern high bridges; this analysis is of primary importance for structural design since a possible failure by elastic instability is unacceptable. Generally these structures show constructive and geometric continuity between the pile and the foundation pile or the part of the pile embedded in the subgrade. A simplified column model is adopted, which includes variation of the section and presence of lateral elastic restriction to model the elastic feature (behavior) of the soil, being able to analyze stratified soil with variables characteristics in depth. AASHTO (American Association of State Highway and Transportation Officials), allows to estimate the fixed length of piles embedded in clay or sand by means of empirical methods. From the weak formulation of the problem, the Galerkin Method was applied to determine the critical loads and the respective buckling lengths.

## 1 INTRODUCTION

The critical buckling loads assessment in column-type structural elements, is of primary importance in structural design. Although the most usual mathematical model in the practice is the bars- model with bending deformation according to Euler-Bernoulli, in this work the deformation by shear stress is considered, resulting in a more complete model. In classical problems, columns can present different support conditions at their ends. To analyze these cases there is a wide bibliography in which the mathematical models and the exact (classic) solutions are correspondingly detailed. In civil engineering for example, the possibility of failure due to buckling in deep foundations (with pilings) can sometimes be underestimated, so it is necessary to take into account in the mathematical model the foundation structure and its interaction with the ground. In the case of slender mechanical elements, thickness reduction is a common designing practice to get lighter pieces, especially when they are moving ones.

The main goal of this work is to determine values of critical buckling loads and their corresponding failure modes in the case of slender piles that make up a structural unit pile foundation-pile, mainly used in intermediate supports of the bridge decks, considering the elastic lateral restriction provided by the soil. Obtaining the weak formulation of the problem allows the application of the Galerkin Method and, thereafter, values of critical loads and buckling modes causing instability of the elastic balance can be assessed. In this first stage, piles are considered to develop only endurance capacity on a resistant support layer, neglecting the lateral friction between the soil and the surface of the embedded pile.

## 2 FORMULATION

Consider the structure shown in Figure (1), subjected to a load  $P$  that keeps a direction parallel to the axis  $\bar{x}$ . This element is constituted by an isotropic and homogeneous material, of inertia moment  $I(\bar{x})$  whose expression is a function of the variable section. The incorporation of a lateral elastic restriction  $K(\bar{x})$  limits the transverse displacements in the embedded length. These displacements are assumed possible only in the plane  $\bar{x} - \bar{y}$ .

Without losing generality in the application of the method and for illustrative purposes, in Figure (1.1) it is considered a problem with a pile tip supported on sturdy stratum ( $\bar{x} = 0$ ) where the point bearing and horizontal frictional capacities develop, the latter preventing lateral movement, and lateral elastic restriction  $0 \leq \bar{x} \leq L_{EMB}$ , with  $L_{EMB}$  being the length of the part embedded in the ground, and restricted lateral displacement at the end ( $\bar{x} = L$ ). The boundary problem in this case considering the shear stress, is:

$$\begin{aligned} \frac{d^2}{d\bar{x}^2} \left[ EI(\bar{x}) \frac{d^2 w(\bar{x})}{d\bar{x}^2} \right] + P \frac{AG}{(AG - \chi P)} \frac{d^2 w(\bar{x})}{d\bar{x}^2} + K(\bar{x}) w(\bar{x}) &= 0, \quad \forall \bar{x} \in (0, L); \\ w(\bar{x})|_{\bar{x}=0} &= 0, \quad \frac{d^2 w(\bar{x})}{d\bar{x}^2} \Big|_{\bar{x}=0} = 0; \quad y \quad w(\bar{x})|_{\bar{x}=L} = 0, \quad \frac{d^2 w(\bar{x})}{d\bar{x}^2} \Big|_{\bar{x}=L} = 0. \end{aligned} \quad (1)$$

Adopting the variable change  $x = \bar{x}/L$ , the boundary problem is stated in dimensionless form as follow:

$$\begin{aligned} \frac{d^2}{dx^2} \left[ \frac{I(x)}{I_0} \frac{d^2 w(x)}{dx^2} \right] + \alpha \frac{d^2 w(x)}{dx^2} + \beta k(x) w(x) &= 0, \quad \forall x \in (0, 1); \\ w(x)|_{x=0} &= 0, \quad \frac{d^2 w(x)}{dx^2} \Big|_{x=0} = 0; \quad y \quad w(x)|_{x=1} = 0, \quad \frac{d^2 w(x)}{dx^2} \Big|_{x=1} = 0. \end{aligned} \quad (2)$$

where  $I_0 = I(x)_{x=0}$ ,  $\alpha = \frac{AG}{(AG - \chi P)} \frac{PL^2}{EI_0}$  and  $\beta = \frac{L^4}{EI_0}$ . The appropriate space to locate the

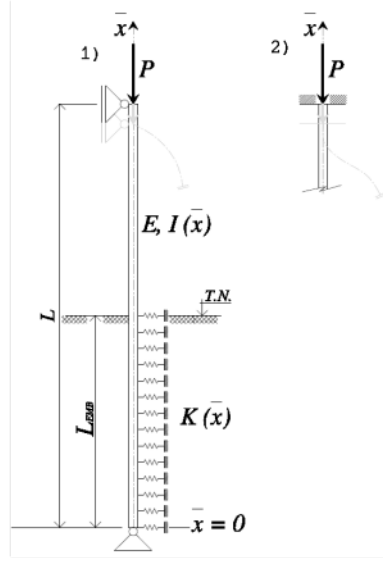


Figure 1: Pile propped on a sturdy ground ( $\bar{x} = 0$ ). 1): Lateral displacement constrained at  $\bar{x} = L$ . 2): Rotation and lateral displacement fixed at  $\bar{x} = L$ .

boundary problem (2) assuming  $k, I \in L^2(\Omega)$ , is:

$$V(\Omega) = \{v \in H^2(\Omega), \quad v|_{x=0} = 0, \quad v|_{x=1} = 0\}. \quad (3)$$

To derive the weak formulation, the usual procedure is applied, i.e.:

$$\int_0^1 \left\{ \frac{d^2}{dx^2} \left[ \frac{I(x)}{I_0} \frac{d^2 w(x)}{dx^2} \right] + \alpha \frac{d^2 w(x)}{dx^2} \right\} v(x) dx + \int_0^{\frac{L_{EMB}}{L}} \beta k(x) w(x) v(x) dx = 0, \quad (4)$$

for all  $v \in V(\Omega)$ , with  $k(x) = 0$  between  $L_{EMB}/L < x \leq 1$ . The application of the integration formula by parts: twice to the first term, once to the second term of the integral in (4), and considering the boundary conditions of the problem, leads to the following expression:

$$\frac{1}{I_0} \int_0^1 I(x) w''(x) v''(x) dx - \alpha \int_0^1 w'(x) v'(x) dx + \beta \int_0^{\frac{L_{EMB}}{L}} k(x) w(x) v(x) dx = 0, \quad (5)$$

Consequently, the weak formulation results:

$$\begin{cases} \text{Find } w \in V(\Omega) \text{ such that} \\ a(w, v) - \alpha (w', v')_{L^2} = 0, \quad \forall v \in V(\Omega), \end{cases} \quad (6)$$

$$\text{where} \quad a(w, v) = \frac{1}{I_0} \int_0^1 I(x) w''(x) v''(x) dx - \alpha \int_0^1 w'(x) v'(x) dx \quad (7)$$

$$\text{and} \quad (w', v')_{L^2} = \beta \int_0^{\frac{L_{EMB}}{L}} k(x) w(x) v(x) dx \quad (8)$$

The formulation (6) is a problem of eigenvalues and the existence and uniqueness of its solution are not addressed in this paper, but they can be proven.

### 3 APPLICATION OF THE GALERKIN METHOD

From the variational formulation obtained in the previous section, it is possible to apply the Galerkin method to obtain approximate values for critical loads dimensionless coefficients  $\alpha$ .

Considering the space of finite dimension  $V^h(0, 1) \subset H^2(0, 1)$ , the weak formulation (6) in this space results:

$$\begin{cases} \text{Find } w_h \in V^h(\Omega) \text{ such that} \\ a(w_h, v_h) - \alpha (w_h', v_h')_{L^2} = 0, \quad \forall v_h \in V^h(\Omega). \end{cases} \quad (9)$$

Any function belonging to  $V^h$  can be expressed as a linear combination of a base, thus  $w_h, v_h$ , result:

$$w_h = \sum_{j=1}^N c_j \varphi_j, \quad v_h = \sum_{i=1}^N d_i \varphi_i, \quad (10)$$

where  $\varphi_i(x)$  are functions of the basis of  $V^h$  and  $N$  the number of functions for a required or desired approximation.

By replacing the expressions (10) in (9) and operating algebraically, the following problem of eigenvalues is obtained:

$$(K - \alpha M) C = 0, \quad (11)$$

where the matrices  $K$  and  $M$  are symmetric, and  $C$  is the vector of the unknown parameters  $c_j$ . The  $K$  and  $M$  components, are obtained in the following way:

$$\begin{aligned} K_{i,j} &= a(\varphi_i, \varphi_j), \\ M_{i,j} &= (\varphi_i', \varphi_j')_{L^2}. \end{aligned} \quad (12)$$

Its application and satisfactory comparison with the results obtained and published for classical problems with other methods (Wang et al., 2005), can be consulted in [Albarracin et al. \(2016\)](#).

### 4 APPLICATION TO THE FOUNDATION SYSTEM OF BRIDGE PILES

The instability of the elastic balance is a condition to be taken into account in the case of batteries (columns), particularly in the case of very slender piles. It is common in high bridges to build individual piers from the foundation pile up to the superstructure. It is proposed to analyze the behavior of the global elastic equilibrium as a continuous pile structure (considering the embedded part and the free one). The pile propped on a sturdy ground is assumed to act as a pinned support with free rotation of the section at  $\bar{x} = 0$  (Figure 1), a condition that can occur in practice.

#### 4.1 Application

Following the boundary problem (2) (Figure 1.1), a comparison is made between the results obtained by applying the Galerkin method with others attained by means of the differential transformation method (DTM) and the analytic solution of the differential equation ([Çatal et al., 2006](#)).

The basis adopted for the space  $V^h$  (see 3) in this boundary problem is:

$$\{\varphi_1(x) = (x-1)x, \quad \varphi_i(x) = \varphi_1(x) + (x-1)x^i; \quad i = 2, 3, \dots, N\}. \quad (13)$$

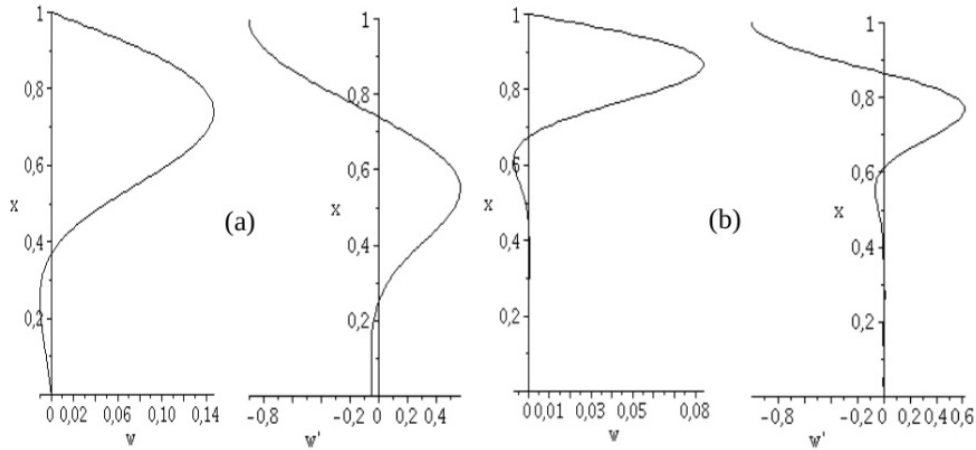


Figure 2: Fundamental buckling mode and derivative function applying Galerkin's method at: (a)  $L = 21,0$  m y  $L_{EMB}/L = 0,5$ ; (b):  $L = 37,35$  m y  $L = 21,0$  m y  $L_{EMB}/L = 0,75$ .

Applying the Galerkin method, the values of the last column of Table 1 are obtained from the data and parameters given in the example, by Çatal et al. (2006), for this boundary problem (Figure 1.1). The results for the first mode and 2 different length configurations are compared: Conf. (a) and (b).

		DTM (Çatal, S. et al., 2006)			Galerkin, $N = 14$
Conf.	$L_{EMB}/L$	$L[m](1)$	$N_{cr}/N_E(2)$	$N_{cr}[MN](3)$	$P_{critico}[MN]$
(a)	0,5	21,00	4,682876	30,5826	30,5827
(b)	0,75	37,35	17,571108	36,2879	36,2508

Table 1: Çatal et al., 2006: (1)Table 1; (2)Table 2(b); (3)from (2) and  $N_E(Euler)$ .

In the Figure 2, the first mode and the derived functions are plotted for both configurations (a) and (b), when applying the Galerkin method. The data and parameters used are taken from (Çatal et al., 2006):  $I = 1.39 \times 10^{-3} m^4$ ;  $E.I = 291.9 MPa$ ;  $A.G = 2053.791 MN$ ;  $\chi = 1/0.4347$ ;  $k(x).b = 15 MPa$ .

## 4.2 Reaction Subgrade Soil Module

To model the subgrade soil through lateral constraints we apply the concept of the soil (or subgrade soil) reaction modulus. The correct estimation of this module is not trivial and it is not exempt from a series of simplifying hypotheses that increase the uncertainties about its evaluation; not a few researchers have worked on estimating the reaction module for different supported structures and others buried in the subsoil: Terzaghi (1955); Davisson et al. (1965); Prakash et al. (1990); Bowles, J.E. (1997). Most of these analyses are based on the study of the elastic modulus of vertical soil or subgrade reaction.

Much of the work that deals with the lateral subgrade reaction in buried structures is derived from the studies of Vesic (1961, 1977), which initially based his analysis, in Winkler's hypothesis: Winkler (1867) and Biot, M.A. (1937). Vesic concludes that a very good approximation to the study of a beam of infinite length, subjected to bending and shear and supported on the subsoil considered as an elastic, homogeneous and isotropic and semi-infinite medium, can be achieved from. Vesic concludes that a very good approximation to the case of a beam of infinite

length, subjected to bending and shear and supported on the subsoil considered as an elastic, homogeneous, isotropic and semi-infinite medium, can be achieved making use of Winklers hypothesis. Winkler postulates that the relation between the contact pressure beam-ground and the beam deflection remains constant throughout its length, so:

$$\frac{p}{w} = ctte = k_o \quad (14)$$

Following the classical theory of a beam supported on an elastic medium, it is assumed that for a beam width  $b$ , the reaction per unit length and unit vertical displacement is  $K = k_o.b$  in MPa. Then for a vertical displacement  $\eta$ , the distributed reaction will be:

$$r = K\eta \quad (15)$$

If the beam is subjected to a distributed load of intensity  $q(\bar{x})$  the total distributed force per unit beam length is  $(q - r = q - K\eta)$ , and the differential equation of the elastic line:

$$E_b I_b \frac{d^4 \eta}{dx^4} = q - K\eta; \quad \text{so:} \quad q(\bar{x}) = 0, \quad \eta^{IV} + 4\lambda^4 \eta = 0. \quad (16)$$

In (16),  $\lambda$  parameter is:

$$\lambda^4 = \frac{K}{4E_b I_b} \quad \text{so} \quad \lambda = \sqrt[4]{\frac{K}{4E_b I_b}} \quad (17)$$

being  $\lambda$ , the inverse of a length.

The general solution of equation (16) (taking into account only the terms damped by physical consistency), will be of the form:

$$\eta = C_1 e^{-\lambda \bar{x}} \sin(\lambda \bar{x}) + C_2 e^{-\lambda \bar{x}} \cos(\lambda \bar{x}) \quad (18)$$

On the other hand, [Biot, M.A. \(1937\)](#) obtains the bending solution for a beam of width  $b$  with a point load  $P$  and also for a concentrated moment  $M$  under the same hypothesis. He defines a parameter  $c$  as the fundamental length of the beam with continuous support:

$$c = \sqrt[3]{C(1 - \mu_s^2) \frac{E_b I_b}{E_s}}, \quad (19)$$

being for the previous one:

$E_b$ : beam Young modulus;

$\mu_b$ : beam Poisson modulus;

$I_b$ : beam inertia modulus;

$E_s$ : subgrade soil Young modulus;

$C$ : dimensionless parameter with values in the range between 1.0 and 1.13. It depends on the distribution of stresses in the width  $b$ , where  $C = 1.0$  for the uniform distribution.

[Vesic \(1961\)](#), following a rigorous procedure, numerically evaluates the integrals in the Biot solution and approximates them as a function of the  $b/c$  ratio, arriving to a damped wave solution with a damping factor:

$$\lambda = \frac{0,689}{b} \left( \frac{b}{c} \right)^{0,813} \quad (20)$$

Vesic points out that this factor can be equated to the damping factor  $\lambda$  given in (17), which is a function of the soil (or subgrade soil) reaction modulus  $K$  and from which its value can be extracted. He proposes as an expression for the soil reaction modulus:

$$k_{\infty}B = K_{\infty} = \frac{0.9}{C} \sqrt[12]{\frac{E_s b^4}{E_b I_b} \frac{E_s}{1 - \mu^2}} \quad (21)$$

Introducing in (21) (Vesic, 1961),  $B = 2b$  and  $C = 1.10$  (this last value considered correct by Vesic for any practical purpose), yields to:

$$K_{\infty} = 0.65 \sqrt[12]{\frac{E_s B^4}{E_b I_b} \frac{E_s}{1 - \mu^2}} \quad (22)$$

From (22) and observing that the lateral soil reaction modulus can vary with depth depending on its elasticity modulus, setting the pile diameter as  $B = D$ , it is possible to adjust in depth the Vesic expression:

$$K_{Vesic}(\bar{x}) = 0.65 \sqrt[12]{\frac{E_s(\bar{x}) D^4(\bar{x})}{E_b I_b(\bar{x})} \frac{E_s(\bar{x})}{1 - \mu^2}}, \quad \text{en MPa.} \quad (23)$$

The expression (23) allows the reaction modulus to be constant or variable. It is of particular interest the  $K$  variation as a function of the depth to vary  $E_s(\bar{x})$  which allows adjusting the value of the reaction modulus based on experimental observations, concerning with the behavior of embedded piles in different types of soils ranging from sands to clays. For purely frictional soils, empirical results or empirical facts drive to admit a linear or quasi-linear variation for the reaction modulus (Bowles, J.E., 1997), while for cohesive soils a constant  $K$  is an appropriate approach (Pochman et al., 1989).

It is proposed to apply a depth-dependent refinement function, which will allow to calibrate (23):

$$E_s(\bar{x}) \cong E_{s(Ref)} \left[ s_o + \left( \frac{\bar{x}}{z_o} \right)^n \right]^{\frac{1}{m}} \quad \text{y} \quad K(\bar{x}) \cong K_{Vesic}(\bar{x}) \quad \text{both in MPa,} \quad (24)$$

whith:

$s_o$ : dimensionless value, being 1 (clays) or 0 (sands), adjusts the elasticity modulus on the land surface;

$E_{s(Ref)}, z_o$ : "reference values" for the soil elasticity modulus and depth;

$\bar{x}$ : subgrade soil embedded length;

$n$ : dimensionless adjustment factor (e.g. 1 for non-cohesive soils);

$m$ : dimensionless adjustment factor (e.g. 1 for non-cohesive soils).

For illustrative purposes, in Figure 3 theoretical curves are shown for the  $E_s(\bar{x})$  variation, obtained with (24), for  $E_{s(Ref)} = 50 \text{ MPa}$ ,  $Z_o = 30 \text{ m}$ , with the refinement values starting from the top quasi-constant curve to the down linear curve, in the figure above: a)  $s_o = 1$ ,  $n = 0.01$ ,  $m = 15.5$ ; b)  $s_o = 0.25$ ,  $n = 0.7$ ,  $m = 5.5$ ; c)  $s_o = 0.1$ ,  $n = 0.3$ ,  $m = 9.0$ ; d)  $s_o = 0.1$ ,  $n = 0.8$ ,  $m = 2.0$ ; e)  $s_o = 0$ ,  $n = 1$ ,  $m = 1$ ;

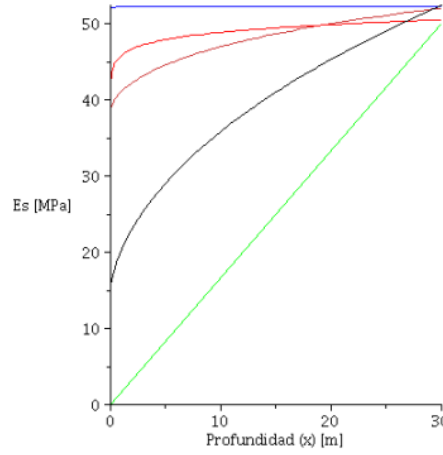


Figure 3:  $E_s$ , Depth-dependent.

#### 4.3 Particular case application: obtaining critical load and fundamental mode with estimation of pile elastic embedment length

Lets take case of Figure (1.2) which shows a pile-bridge stack structure fixed in  $\bar{x} = L$ . The pile, of circular section and diameter  $D$ , is embedded in a standard resistance clay with:  $E_{s(Ref)} = 50MPa$ , AASHTO (2017), Section 10, Table 10.6.2.2.3b-1, (Bowles, J.E., 1988); and a total length  $L = 20m$ , being the embedded pile length  $L_{EMB} = 10m$ , and at  $\bar{x} = 0$  the tip is supported by a resistant soil layer (Figure 1).

To obtain expressions in dimensionless form, we adopt  $x = \bar{x}/L$  resulting:

$$k_1(x) = K_{Vesic} \left(1 - \frac{\bar{x}}{L}\right) = K_{Vesic} (1 - x), \quad k_2(x) = 0. \quad (25)$$

With (25) we propose to apply at (7), the following elastic restrictions:

$$k(x) = \begin{cases} 0.65 \sqrt[12]{\frac{E_s(1-x)^{64.0}}{\pi E_b} \frac{E_s(1-x)}{(1-\mu^2)}}; & 0 \leq x \leq 1/2 \\ 0; & 1/2 < x \leq 1 \end{cases} \quad (26)$$

The boundary problem (Figure 1.2) in dimensionless form results:

$$\begin{aligned} \frac{d^2}{dx^2} \left[ \frac{I(x)}{I_0} \frac{d^2 w(x)}{dx^2} \right] + \alpha \frac{d^2 w(x)}{dx^2} + \beta k(x) w(x) &= 0, \quad \forall x \in (0, 1); \\ w(x)|_{x=0} &= 0, \quad \frac{d^2 w(x)}{dx^2} \Big|_{x=0} = 0; \quad y \quad w(x)|_{x=1} = 0, \quad \frac{dw(x)}{dx} \Big|_{x=1} = 0; \end{aligned} \quad (27)$$

an analogous procedure as the previous case, leads to the same expressions (6), (7) y (8), being the appropriate space to locate this new problem, assuming  $k, I \in L^2(\Omega)$ :

$$V(\Omega) = \{v \in H^2(\Omega), \quad v|_{x=0} = 0, \quad v|_{x=1} = 0, \quad v'|_{x=1} = 0\}. \quad (28)$$

The adopted basis for the  $V^h$  space, is:

$$\{\varphi_1(x) = x/6 - x^3/2 + x^4/3, \quad \varphi_i(x) = \varphi_1(x) + (x-1)^2 x^{(i+2)}; \quad i = 2, 3, \dots, N\}. \quad (29)$$

The expression (23) is adjusted through the (24), with:  $Z_o = 30m$ ;  $s_o = 1$ ,  $n = 0.1$ ,  $m = 15.5$ ,  $\mu = 0.3$ . Figure (3) (left) shown of  $k(x)$  depth-dependent (ec. 26). The remaining



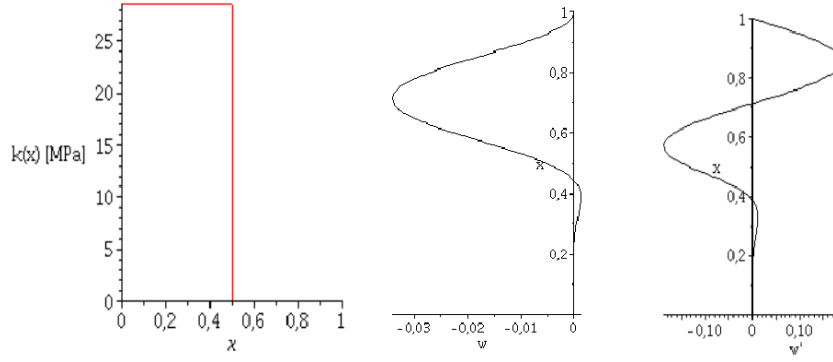


Figure 4: Reaction Subgrade Soil Modulus  $k(x)$  (left), fundamental mode (center) and derivative function (right).

data for the example are:  $D_{pilote} = D_b = 0.40m$ ;  $\mu_b = 0.2$ ;  $E_b = 4700\sqrt{f'_c}MPa$ , with  $f'_c = 25MPa$ .

A summary of the results obtained for the first mode with  $L = 20m$  and  $L_{EMP}/L = 0.5$ , applying the Galerkin method with the space given by (28) and the basis adopted at (29) is shown in Table 2.

Galerkin	$N = 20$	$N = 30$
$P_{critic}$ [MN]	8,79497	8,79359
$L_{free}$ [m]	12,275	12,295
$L_{empot}$ [m]	7,725	7,705

Table 2: Critical load and characteristic lengths for the fundamental buckling mode.

At Figure 4 (center and right), the fundamental buckling mode and the derivative function are plotted respectively. In Table 2, the critical buckling load, the free buckling length and of pile elastic embedment length  $L_{empot}$ , are transcribed.  $L_{empot}$  is the embedded length from  $x = 0$  to the pile rotationless section ( $w' = 0$ ), closest to the natural terrain surface (or bed). From the position of this section we approach the free buckling length as:  $L - L_{empot}$ .

The AASHTO (2017) specifications in its section 10 -10.7.4.2 -(Davisson et al., 1965)-, allows the estimation of the depth from which the pile can be considered elastic embedment in clay, applying  $1.4(E_p I_p / E_s)^{0.25}$ . While with the Galerkin method we obtain an elastic embedding length  $L_{empot} \simeq 7.70m$  (last line in the Table 2), with this last expression (AASHTO, 2017) the elastic embedment length is estimated at  $8.77m$ .

## 5 CONCLUSIONS

The Galerkin method is simple to implement in computers, and without losing flexibility for the analysis of different problems, it can be used in those of interest in applied engineering. In practice, the estimation of the soil reaction modulus is difficult and the expression proposed by Vesic (1961) is a good alternative for the project stage. It is advisable during the executive design, to obtain experimental and laboratory values to generate and calibrate the appropriate curves for the soil layers under study. In the last example, the elastic embedment length obtained with the Galerkin method and that obtained through 10.7.4.2-1 of the AASHTO (2017) are shown, noticing that this empirical expression is limited to considering the stratum close to the surface. Once the values and parameters have been refined during the executive design, the Galerkin method provides the necessary data for the structural design, such as the buckling

length and the critical load, while the link conditions can be modeled to represent the physical reality of the structure. As next goals, the extension of the model for simultaneous axial and lateral loads with elastic restrictions at the ends and lateral friction in the embedded pile, which would allow modeling and addressing more general situations to those shown in this work, are envisaged.

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